

Bounds on the phase velocity in the linear instability of viscous shear flow problem in the β -plane

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Abstract. Results obtained by Joseph (*J. Fluid Mech.* **33** (1968) 617) for the viscous parallel shear flow problem are extended to the problem of viscous parallel, shear flow problem in the beta plane and a sufficient condition for stability has also been derived.

Keywords. Viscous shear flows; linear stability.

1. Introduction

Parallel shear flows problem is a classical hydrodynamic instability problem and continues to attract attention of researchers [1,2,3,5]. Kuo [8] considered slightly general case of a homogeneous, inviscid, parallel shear flow problem in the β -plane. He obtained the linearized perturbation equation for this problem and derived an extension of the celebrated Rayleigh's [9] inflexion point criterion for inviscid, homogeneous parallel shear flows. Fjortoft's [4] instability criterion for inviscid, homogeneous parallel shear flows has also been extended for such barotropic flows (see [6]). Hickernell reviewed various results obtained for this problem and obtained new upper bound on the growth rate of the temporally growing modes of the linearized equations of motion for this problem. Joseph [7] obtained bounds for the eigenvalues of the Orr–Sommerfeld's equation. Derivation of bounds is of importance because exact solution of the problem is not obtainable in closed form. In the present paper, we extend the results of Joseph [7] to the problem of viscous parallel shear flow problem in the β -plane and also obtain a sufficient condition for stability.

The governing equation for Kuo's [8] problem for the linear stability of an incompressible, viscous, parallel shear flow in the β -plane is given by (cf. [6])

$$(U - c)(D^2 - \alpha^2)\phi - (D^2U - \beta)\phi = \frac{(D^2 - \alpha^2)^2}{i\alpha R}\phi \quad (1)$$

and the associated boundary conditions are that ϕ must vanish on the rigid walls which may recede to $\pm\infty$ in the limiting cases and thus

$$\phi = D\phi = 0 \quad \text{at} \quad z = -1 \quad \text{and} \quad z = 1, \quad (2)$$

where z is the real independent variable such that $-1 \leq z \leq 1$, $D \equiv d/dz$, α is a real constant and denotes the wave number, $c = c_r + ic_i$ is the complex wave velocity, $U(z)$ is a

twice continuously differentiable function of z and denotes the prescribed basic velocity distribution while the dependent variable $\phi(z)$ is, in general, a complex valued function of z and denotes the z component of velocity distribution of the parallel flow, the parameter β is the derivative of the Coriolis force in latitudinal direction [6] and R is the Reynolds number. Here, we note that eq. (1) for $\beta = 0$ reduces to the celebrated Orr–Sommerfeld’s equation.

2. Mathematical analysis

We prove the following theorems:

Theorem 1. *If (ϕ, c, α^2) is a solution of eigenvalue problem prescribed by eqs (1) and (2) for given values of R and β , then*

$$I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2 = i\alpha c R (I_1^2 + \alpha^2 I_0^2) - i\alpha R Q, \quad (3)$$

where we take

$$I_n^2 = \int_{-1}^1 |D^n \phi|^2 dz \quad (n = 0, 1, 2) \quad (4)$$

and

$$\begin{aligned} Q = \int_{-1}^1 \{U(|D\phi|^2 + \alpha^2 |\phi|^2) + (D^2 U - \beta)|\phi|^2\} dz \\ + \int_{-1}^1 DU(D\phi)\phi^* dz. \end{aligned} \quad (5)$$

Proof. Multiplying eq. (1) by ϕ^* (the complex conjugate of ϕ) throughout and integrating the resulting equation over the vertical range of z , using the boundary conditions (2), we derive

$$I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2 = i\alpha c R (I_1^2 + \alpha^2 I_0^2) - i\alpha R Q. \quad (6)$$

This proves the theorem.

Theorem 2. *If (ϕ, c, α^2) is a solution of eigenvalue problem prescribed by eqs (1) and (2) for given values of R and β , then*

$$c_r = \frac{\text{Re}(Q)}{I_1^2 + \alpha^2 I_0^2} \quad (7)$$

and

$$c_i = \frac{1}{(I_1^2 + \alpha^2 I_0^2)} \left\{ \text{Im}(Q) - \frac{(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2)}{\alpha R} \right\}, \quad (8)$$

where

$$\text{Re}(Q) = \int_{-1}^1 \left\{ U(|D\phi|^2 + \alpha^2 |\phi|^2) + \left(\frac{D^2 U}{2} - \beta \right) |\phi|^2 \right\} dz \quad (9)$$

and

$$\text{Im}(Q) = \frac{i}{2} \int_{-1}^1 \{DU \{ \phi (D\phi^*) - (D\phi)\phi^* \} \} dz. \quad (10)$$

Proof. Equating the real part of both sides of eq. (3), we obtain

$$I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2 = -\alpha c_i R (I_1^2 + \alpha^2 I_0^2) + \alpha R \operatorname{Im}(Q)$$

or

$$c_i = \frac{1}{(I_1^2 + \alpha^2 I_0^2)} \left\{ \operatorname{Im}(Q) - \frac{(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2)}{\alpha R} \right\}. \quad (11)$$

Now, equating the imaginary parts of both sides of eq. (3), we obtain

$$c_r = \frac{\operatorname{Re}(Q)}{(I_1^2 + \alpha^2 I_0^2)}. \quad (12)$$

This proves the theorem.

Theorem 3. If (ϕ, c, α^2) is a solution of eigenvalue problem prescribed by eqs (1) and (2) for given values of R and β , then

$$c_i \leq \frac{q}{2\alpha} - \frac{1}{\alpha R} \left(\frac{\pi^2}{4} + \alpha^2 \right), \quad (13)$$

where $q = \max |DU|$ on $[-1, 1]$.

Proof. Taking modulus of both sides of eq. (10), we obtain

$$\begin{aligned} \operatorname{Im}(Q) &\leq |\operatorname{Im}(Q)| = \left| \frac{i}{2} \int_{-1}^1 DU \{ \phi (D\phi^*) - (D\phi) \phi^* \} dz \right| \\ &\leq \int_{-1}^1 |DU| |\phi| |D\phi| dz \leq q \int_{-1}^1 |\phi| |D\phi| dz \\ &\leq q \sqrt{\int_{-1}^1 |D\phi|^2 dz} \sqrt{\int_{-1}^1 |\phi|^2 dz} = q I_1 I_0. \end{aligned} \quad (14)$$

(On using Schwartz's inequality)

On using inequality (14) in eq. (11), we get

$$c_i \leq \frac{1}{(I_1^2 + \alpha^2 I_0^2)} \left\{ q I_1 I_0 - \frac{(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2)}{\alpha R} \right\}. \quad (15)$$

Clearly

$$I_1^2 + \alpha^2 I_0^2 \geq 2\alpha I_1 I_0. \quad (16)$$

Using the isoperimetric inequalities $I_2^2 \geq (\pi^2/4) I_1^2$ and $I_1^2 \geq (\pi^2/4) I_0^2$, we have

$$\begin{aligned} I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2 &= (I_2^2 + \alpha^2 I_1^2) + \alpha^2 (I_1^2 + \alpha^2 I_0^2) \\ &\geq \left(\frac{\pi^4}{4} + \alpha^2 \right) I_1^2 + \alpha^2 \left(\frac{\pi^4}{4} + \alpha^2 \right) I_0^2 \\ &= \left(\frac{\pi^4}{4} + \alpha^2 \right) (I_1^2 + \alpha^2 I_0^2). \end{aligned} \quad (17)$$

Now, from inequalities (15), (16) and (17), we get

$$c_i \leq \frac{q}{2\alpha} - \frac{1}{\alpha R} \left(\frac{\pi^2}{4} + \alpha^2 \right). \quad (18)$$

This proves theorem 3.

Inequality (18) gives an upper bound on the growth (or decay) rate of a disturbance of wave number α .

Theorem 4. *If (ϕ, c, α^2) is a solution of eigenvalue problem prescribed by eqs (1) and (2) for given values of R and β , and if αR is small enough then the flow will be stable.*

Proof. Inequality (15) can be written as

$$(I_1^2 + \alpha^2 I_0^2) \alpha R c_i \leq \{ \alpha R q I_1 I_0 - (I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2) \}.$$

Thus if αR is small enough then c_i will be negative and the flow will be stable.

Theorem 5. *If (ϕ, c, α^2) is a solution of eigenvalue problem prescribed by eqs (1) and (2) for given values of R and β then the following inequalities hold good:*

(i) *If $D^2 U_{\min}/2 \geq \beta$, then*

$$U_{\min} < c_r < U_{\max} + \frac{4 \left(\frac{D^2 U_{\max}}{2} - \beta \right)}{\pi^2 + 4\alpha^2}. \quad (19)$$

(ii) *If $D^2 U_{\min}/2 \leq \beta \leq D^2 U_{\max}/2$, then*

$$U_{\min} + \frac{4 \left(\frac{D^2 U_{\min}}{2} - \beta \right)}{\pi^2 + 4\alpha^2} < c_r < U_{\max} + \frac{4 \left(\frac{D^2 U_{\max}}{2} - \beta \right)}{\pi^2 + 4\alpha^2}. \quad (20)$$

(iii) *If $D^2 U_{\max}/2 \leq \beta$, then*

$$U_{\min} + \frac{4 \left(\frac{D^2 U_{\min}}{2} - \beta \right)}{\pi^2 + 4\alpha^2} < c_r < U_{\max}, \quad (21)$$

where f_{\max} and f_{\min} are respectively the $\max f(z)$ and $\min f(z)$ on $z \in [-1, 1]$.

Proof. From eq. (7) we have

$$U_{\min} + \frac{\left(\frac{D^2 U_{\min}}{2} - \beta \right) I_0^2}{(I_1^2 + \alpha^2 I_0^2)} < c_r < U_{\max} + \frac{\left(\frac{D^2 U_{\max}}{2} - \beta \right) I_0^2}{(I_1^2 + \alpha^2 I_0^2)}. \quad (22)$$

Now, we prove part (i) of the theorem. Since, $D^2 U_{\min}/2 \geq \beta$, therefore

$$U_{\min} < U_{\min} + \frac{\left(\frac{D^2 U_{\min}}{2} - \beta \right) I_0^2}{(I_1^2 + \alpha^2 I_0^2)} < c_r. \quad (23)$$

By Rayleigh–Ritz inequality, we have

$$I_1^2 \geq \frac{\pi^2}{4} I_0^2 \quad (\text{since } \phi(-1) = \phi(1) = 0). \quad (24)$$

Using inequality (24), we obtain

$$\frac{I_0^2}{I_1^2 + \alpha^2 I_0^2} \leq \frac{4}{\pi^2 + 4\alpha^2}. \quad (25)$$

Therefore

$$\frac{\left(\frac{D^2 U_{\max}}{2} - \beta\right) I_0^2}{I_1^2 + \alpha^2 I_0^2} \leq \frac{4 \left(\frac{D^2 U_{\max}}{2} - \beta\right)}{\pi^2 + 4\alpha^2} \quad (26)$$

since, $\beta \leq D^2 U_{\min}/2 \leq D^2 U_{\max}/2$. Therefore

$$c_r < U_{\max} + \frac{\left(\frac{D^2 U_{\max}}{2} - \beta\right) I_0^2}{I_1^2 + \alpha^2 I_0^2} \leq U_{\max} + \frac{4 \left(\frac{D^2 U_{\max}}{2} - \beta\right)}{\pi^2 + 4\alpha^2}. \quad (27)$$

On using (23) and (27), we obtain

$$U_{\min} < c_r < U_{\max} + \frac{4 \left(\frac{D^2 U_{\max}}{2} - \beta\right)}{\pi^2 + 4\alpha^2}. \quad (28)$$

This completes the proof of part (i).

In case (ii), we have $D^2 U_{\min}/2 \leq \beta \leq D^2 U_{\max}/2$, thus $(D^2 U_{\min}/2) - \beta \leq 0$. Therefore, from the inequality (25) we have

$$\frac{\left(\frac{D^2 U_{\min}}{2} - \beta\right) I_0^2}{I_1^2 + \alpha^2 I_0^2} \geq \frac{4 \left(\frac{D^2 U_{\min}}{2} - \beta\right)}{\pi^2 + 4\alpha^2} \quad (29)$$

or

$$U_{\min} + \frac{\left(\frac{D^2 U_{\min}}{2} - \beta\right) I_0^2}{I_1^2 + \alpha^2 I_0^2} \geq U_{\min} + \frac{4 \left(\frac{D^2 U_{\min}}{2} - \beta\right)}{\pi^2 + 4\alpha^2}. \quad (30)$$

Now, using inequalities (22) and (30) we get

$$U_{\min} + \frac{4 \left(\frac{D^2 U_{\min}}{2} - \beta\right)}{\pi^2 + 4\alpha^2} < c_r. \quad (31)$$

On multiplying inequality (25) throughout by $((D^2 U_{\max}/2) - \beta)(\geq 0)$, adding U_{\max} to the resulting inequality throughout and using inequalities (22) and (31) we get

$$U_{\min} + \frac{4 \left(\frac{D^2 U_{\min}}{2} - \beta\right)}{\pi^2 + 4\alpha^2} < c_r < U_{\max} + \frac{4 \left(\frac{D^2 U_{\max}}{2} - \beta\right)}{\pi^2 + 4\alpha^2}. \quad (32)$$

This proves part (ii) of the theorem.

In case (iii) we have $((D^2U_{\max}/2) - \beta) \leq 0$, therefore

$$U_{\max} + \frac{\left(\frac{D^2U_{\max}}{2} - \beta\right) I_0^2}{I_1^2 + \alpha^2 I_0^2} \leq U_{\max}. \quad (33)$$

On multiplying inequality (25) throughout by $((D^2U_{\min}/2) - \beta) (\leq 0)$, adding U_{\min} throughout to the resulting inequality and using inequalities (22) and (33) we obtain

$$U_{\min} + \frac{4\left(\frac{D^2U_{\min}}{2} - \beta\right)}{\pi^2 + 4\alpha^2} < c_r < U_{\max}. \quad (34)$$

This establishes part (iii) of the theorem and this completes the proof of Theorem 5. It is to be noted here that for $\beta = 0$ in eq. (1), we get the eigenvalue problem considered by Joseph [7] and in this case results of Theorem 5 reduce to the results obtained by Joseph.

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